Self-consistent Calculations using Alpha-Alpha Interactions for $^{16}\text{O}$ Nuclei

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ABSTRACT

Using the modified Woods-Saxon potential and Ali-Bodmer potential of the alpha-alpha interactions, a self-consistent calculations are carried out to describe some properties of $^{16}\text{O}$ nuclei such as binding energies, root mean square radii and form factors. A computer simulation search program has been introduced to solve this problem.

The Hilbert space was restricted to three-dimensional variational space spanned by single particle spherical harmonic oscillator orbits. A comparison using other potentials are also considered. A good fit with experimental data was obtained.

Key Words : Nuclear Structure / Self consistance field

INTRODUCTION

The alpha-alpha potentials that have been constructed from the experimental phase shifts show some common features : The alpha-alpha potential is l-dependent but is independent of the incident energy. The ranges of the inner repulsive part and the outer attractive part are of order of two and five fermi, respectively. The repulsive part becomes weaker for higher l while the attractive part becomes stronger. Thus the phenomenological analyses of alpha-alpha scattering established beyond doubt that a static alpha-alpha potential common to all l does not exist. Nevertheless, the attractive part may be taken as common to all l.

The l-dependence enters through the repulsive part and may be regarded as a simple form of velocity dependent. In fact, Ali and Bodmer have suggested a procedure for the construction of such potentials in a fairly unique way. Thus studies of the fundamental model of the alpha-alpha interaction are extremely useful not only for understanding the essential features of the interaction but also for making the qualitative features of the phenomenological alpha-alpha potentials meaningful.

There have been quit a few attempts to describe the alpha-alpha interaction in terms of a long range attractive potential with a repulsive core. As a results the short range repulsive potential was taken to be l-dependent while the long range part was only in exceptional case allowed to vary with l. The core itself was either taken to correspond to a hard core or to provide a soft repulsion only, so a modified Woods-Saxon distribution was assigned to the repulsive potential, where a purely real attractive potential together with a hard or soft repulsion has been shown to lead to consistent results. It turned out to be capable of reproducing the static and dynamic properties of alpha cluster.

In the present work, we use the modified Woods-Saxon potential and Ali-Bodmer potential with the $T_d$ symmetry assigned to $^{16}\text{O}$ nuclei. The $^{16}\text{O}$ nucleus can be described by four alpha cluster arranged on a tetrahedron. As the alpha-particle are bosons, completely symmetric functions have to
be chosen and HF-equations are solved by iteration until self-consistent \(^{(1)}\) is achieved. Binding energies, root mean square radii and form factors are calculated using the modified Woods-Saxon potential and comparison with Gaussian potentials is discussed.

THE THEORY

We assume that the Hamiltonian \( H \) consists of a one-body and a two-body part

\[
H = \hat{T} + \hat{V} = \sum_{i=1}^{N} H_1(i) + 1/2 \sum_{i \neq j=1}^{N} H_2(i, j)
\]  

(1)

Usually, \( T \) is just the kinetic energy and \( V \) the two-body interaction. In Hartree-Fock method, one takes for the best choice of the normalized wave function \( \psi \) the one that it minimizes the expectation value of the Hamiltonian \( H \)

\[
\delta \langle\psi|H|\psi\rangle = 0
\]  

(2)

In most Hartree-Fock calculations for light nuclei one has taken the subspace spanned by the lowest harmonic oscillator shell \( 1 \lambda, \ldots, N \). We assume that all the particles occupy the same orbital \( \lambda \) belonging to the average field. Hence the intrinsic state of the whole system would be described by the symmetric wave function

\[
\psi(1, 2, \ldots, N) = \lambda(1) \lambda(2) \lambda(3) \ldots \lambda(N)
\]  

(3)

In this subspace, the HF orbitals \( \lambda \) are then determined by their expansion coefficients \( m_{\lambda} \).

\[
|\lambda\rangle = \sum_{j=1}^{N} m_{\lambda}^{j} |j\rangle
\]  

(4)

And the HF-Hamiltonian \( h(\lambda_1, \lambda_2, \ldots, \lambda_N) \) is replaced by the matrix

\[
\langle i|h|j\rangle = \langle i|t|j\rangle + \sum_{k} \sum_{l} \sum_{s} \langle i|k|\lambda\rangle \langle \lambda|l|j\rangle m_{l}^{\lambda} m_{i}^{k}
\]  

(5)

Where

\[
\langle i|k|\lambda\rangle = \langle i|k|\lambda\rangle + \langle i|k|\lambda\rangle
\]  

(6)

The HF equations are then replaced by the matrix equation

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One proceeds by iteration until self-consistency is achieved.

**Modified Woods-Saxon Potential**

The effective potential was taken to have the form of an $l$-dependent hard wall plus an attractive potential being constant in respect to different momenta. For the attractive long range part, the usual Woods-Saxon form has been adopted. The complete interaction is given by

\[
V(r) = \{ - V_0 [1 + \exp(r - R_0)/a]^{-1} \} \quad \text{for } r > R_c
\]
\[
= \infty \quad \text{for } r \leq R_c
\]

(8)

The variable $r$ and the parameters $V_0$, $R_0$ and $a$ have their usual meaning and $R_c$ stands for the radius of the hard core. A modified Woods-Saxon distribution was assigned to the repulsive potential. The form of the attractive part remained the same as before. For the sake of completeness, the following equation gives the full interaction:

\[
V(r) = - V_0 \{ [1 + \exp(r - R_0)/a]^{-1} - \alpha [1 + \exp(r - R_c)/a_c]^{-1} \}
\]

(9)

The parameters employed in the computations are $R_0=2.4$ fm, $a=0.5$ fm, $V_0=53.8$ MeV, $R_c=2.1$ fm, $a_c=0.1$ fm, and $\alpha=3$.

Thus the attractive potential is seen to be shallower than in the previous case. The depth of the repulsive part had to be chosen to be three times deeper than $V_0$. At the same time radius and especially diffuseness of the repulsive part are smaller than the ones for the real part.

**Gaussian Potentionals**

The nuclear interaction potentials considered here were constructed by Ali and Bodmer(8). They are angular–momentum dependent and are given by superposition of repulsive and attractive gaussians, namely,

\[
V_{\text{ext}}(r) = V_R \exp(-\mu_R^2 r^2) - V_A \exp(-\mu_A^2 r^2)
\]

(10)

where $(V_R, \mu_R)$ and $(V_A, \mu_A)$ are the depth and inverse ranges of the repulsive and attractive parts, respectively. They are phenomenological potentials in the sense that their parameters are fitted in order to reproduce the experimental phase shifts associated with $\alpha-\alpha$ scattering. They reproduce $\delta_0$, $\delta_2$, and $\delta_4$. They used the potentials $(d_0, d_2, d_4)$ and $(e_0, e_2, e_4)$ whose parameters are listed in Table (1).
Table(1)  The depth ($V_A,\mu_A$) and inverse ranges ($V_R,\mu_R$) of the repulsive and attractive parts with the potentials parameters ($d_0, d_2, d_4$) and ($e_0, e_2, e_4$) are listed.

<table>
<thead>
<tr>
<th></th>
<th>$V_A$(MeV)</th>
<th>$\mu_A$(fm$^{-1}$)</th>
<th>$V_R$(MeV)</th>
<th>$\mu_R$(fm$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_0$</td>
<td>-130</td>
<td>0.475</td>
<td>475</td>
<td>0.7</td>
</tr>
<tr>
<td>$d_2$</td>
<td>-130</td>
<td>0.475</td>
<td>320</td>
<td>0.7</td>
</tr>
<tr>
<td>$d_4$</td>
<td>-130</td>
<td>0.475</td>
<td>10</td>
<td>0.7</td>
</tr>
<tr>
<td>$e_0$</td>
<td>-150</td>
<td>0.5</td>
<td>1050</td>
<td>0.8</td>
</tr>
<tr>
<td>$e_2$</td>
<td>-150</td>
<td>0.5</td>
<td>640</td>
<td>0.8</td>
</tr>
<tr>
<td>$e_4$</td>
<td>-150</td>
<td>0.5</td>
<td>0</td>
<td>-</td>
</tr>
</tbody>
</table>

Solutions in a Three-Dimensional Space

We consider solutions in a three dimensional variational space spanned by the orthonormal states $|1\rangle, |2\rangle$ and $|3\rangle$. In this case, we have twenty-one different symmetrized two-body matrix elements. A HF-orbital $\lambda$ will have the general form

$$|\lambda\rangle = \sum_{j=1}^3 m_j^\lambda |j\rangle$$

(11)

Where

$$\sum_j m_j^\lambda m_j^{\lambda'} = \delta_{\lambda\lambda'}, \quad \sum_\lambda m_j^\lambda m_j^{\lambda'} = \delta_{jj'},$$

(12)

And assume that the coefficient $m_j$'s are real.

To investigate the HF-solutions, we have to specify the alpha-alpha potential. The specific combinations chosen as the basic states depend on the symmetry of the intrinsic structure that is expected from the molecular alpha-particle model. Now we chose basic states which are invariant with respect to the transformation of the symmetry group $T_d$. Therefore, we chose our three basic states as
Where $\eta$ is a parameter determined from the $T_d$ symmetry. Here the oscillator shell model wave functions are given by

$$\beta_0 = (\hbar/m_{\alpha}\omega)^{1/2}$$

is the oscillator parameter.

Matrix Elements of the Kinetic and Potential Energy

Considering HF-solutions with mixed parity (where it is important in nuclear calculations), therefore we have only twelve non-vanishing two-body matrix elements for the potential energy $v_{ijkl}$ and three matrix elements of kinetic energy namely $t_{11}$, $t_{22}$ and $t_{33}$. Therefore the non-vanishing HF-Hamiltonian will be

$$h_{11} = t_{11} + 1/2(N - 1) \left[ v_{1111} m_1^2 + v_{1212} m_2^2 + v_{1313} m_3^2 \right]$$

$$h_{12} = h_{21} = 1/2(N - 1) \left[ (v_{1122} + v_{1221}) m_1 m_2 + (v_{1223} + v_{1322}) m_3 m_2 \right]$$

$$h_{13} = h_{31} = 1/2(N - 1) \left[ (v_{1223} + v_{1322}) m_2 m_3 + (v_{1133} + v_{1313}) m_1 m_3 \right]$$

$$h_{23} = h_{32} = 1/2(N - 1) \left[ (v_{2233} + v_{2322}) m_2 m_3 + (v_{2231} + v_{2321}) m_1 m_3 \right]$$

$$h_{22} = t_{22} + 1/2(N - 1) \left[ v_{2121} m_1^2 + v_{2222} m_2^2 + v_{2323} m_3^2 + 2v_{2123} m_1 m_3 \right]$$

$$h_{33} = t_{33} + 1/2(N - 1) \left[ v_{3131} m_1^2 + v_{3232} m_2^2 + v_{3333} m_3^2 + 2v_{3133} m_1 m_3 \right]$$
NUCLEAR DENSITY AND FORM FACTOR

The parity projected part of the nuclear density distribution, normalized to unity, which corresponds
to a HF-solution as defined by Eqs. (13-15), is found to be

\[ \rho_\alpha(r) = \left(1/\beta_0^3 \pi^{2/3}\right) \left[m_1^2 + (8/105)(m_2^6/\beta_0^6)(m_2^2 + 2m_3^2r^2/9\beta_0^2)\right] \exp(-r^2/\beta_0^2) \]  

The form factor corresponding to the spherical part of the density distribution (24) is

\[ F_0(q) = \left[1 - q^2 \beta_0^2 (3m_2^2 + 4m_3^2) / 6 + q^4 \beta_0^4 (m_2^2 + 2m_3^2) / 120 - q^6 \beta_0^6 (m_2^2 + 4m_3^2) / 840 + q^8 \beta_0^8 m_3^2 / 15120\right] \exp(-q^2 \beta_0^2 / 4) \]  

where q is the momentum transfer. The expression of \( F_0(q) \) has to be multiplied by a form factor
\( F(q) = \exp(-q^2 \beta_\alpha^2 / 4) \), which account for the \( \alpha \)-particle distribution.

NUMERICAL RESULTS AND DISCUSSION

Calculations indicated that the following sets of values would give the best overall agreement with
the ground state properties of \(^{16}\text{O}\) using \( \eta = 1.0 \) in Eq. (15). The parameters used are given in table (2).

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>( V_0 (\text{MeV}) )</th>
<th>( R_0 (\text{fm}) )</th>
<th>( A (\text{fm}) )</th>
<th>( R_c (\text{fm}) )</th>
<th>( a_c (\text{fm}) )</th>
<th>( \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(^{16}\text{O})</td>
<td>78</td>
<td>2.4</td>
<td>0.5</td>
<td>2.1</td>
<td>0.1</td>
<td>2.064</td>
</tr>
</tbody>
</table>

And in comparison with the Gaussian potential, the parameters used are given in table (3).

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>( V_A (\text{MeV}) )</th>
<th>( \mu_A (\text{fm}^{-1}) )</th>
<th>( V_R (\text{MeV}) )</th>
<th>( \mu_R (\text{fm}^{-1}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(^{16}\text{O})</td>
<td>130</td>
<td>0.5</td>
<td>260</td>
<td>0.8</td>
</tr>
</tbody>
</table>
The binding energies and root mean square radii are given in Table (4).

**Table (4)** The calculated binding energies and root mean square radii of $^{16}$O using the modified Woods-Saxon potential and Gaussian potential

<table>
<thead>
<tr>
<th>Potentials:</th>
<th>Woods-Saxon Potential</th>
<th>Gaussian Potential</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Theor. rms(fm)</td>
<td>Theor. Rms(fm)</td>
</tr>
<tr>
<td>$^{16}$O</td>
<td>-14.314</td>
<td>-12.970</td>
</tr>
<tr>
<td></td>
<td>2.728</td>
<td>2.64</td>
</tr>
</tbody>
</table>

Fig 1: charge form factors of $^{16}$O nuclei using the Gaussian potential

In Fig.(1), we have plotted the charge form factors of $^{16}$O by using the Gaussian potential without any correction and using the second set of parameters given in table (3) which give good results of binding energies and root mean square radii comparing with experimental results(-14.40 for binding energy and 2.71 for root mean square radius)\(^{(13)}\). Also, the form factor gives a reasonable results comparing with the experimental elastic scattering charge form factor \(^{(13)}\), and with the previous theoretical curves; HO (harmonic oscillator potential), HO+SRC (short range correlation) and HF (Hartree fock using Skyrme – type wave function).
The charge form factors of $^{16}$O nuclei using the modified Woods-Saxon potential without any correction and using the parameters given in table (2) which also give good results of binding energies and root mean square radii (Table 4).

It was found that, in case of $^{16}$O nucleus, we have good agreement with experimental values up to the second minimum and also with the other theoretical cases. Concluding, it can be said that the results presented so far provide some support for this modified potential. In the very end, a soft repulsion is believed to give rise to a more realistic description.

**Fig 2: charge form factors of $^{16}$O nuclei using the modified Woods-Saxon potential**

REFERENCES